Three Dimensional Integrable Mappings

Apostolos Iatrou *†‡

Abstract

We derive three-dimensional integrable mappings which have two invariants.

1 Introduction

In this paper we focus on three-dimensional integrable autonomous mappings preserving at least one three-quadratic (possibly rational) integral (we considered the four-dimensional case (together with its generalization to higher dimensions) in [6]). A major reason for such a study is the lack off results on three-dimensional integrable mappings. A recent paper, which makes some progress on three-dimensional integrable mappings, is [5]. In this paper Hirota *et al* used algebraic entropy [4] to determine which three-dimensional mappings of a particular form had polynomial growth implying zero algebraic entropy. Having discovered all such possible mappings, they used a procedure outlined in their paper to find two functionally independent conserved quantities for each map. In this paper we will take a different approach to the one used by Hirota *et al* to construct three-dimensional integrable mappings. For the purposes of this paper, we consider a three-dimensional autonomous mapping integrable if there exist two functionally independent integrals in involution with respect to some Poisson structure.

The plan of this paper is as follows : In Section 2 we derive three-dimensional volume-preserving mappings which preserve a three-quadratic expression (a method introduced in [2] on a rational four-quadratic expression), then assuming that these three-dimensional volume-preserving mappings have a second integral with a particular ansatz we find 3 three-dimensional volume-preserving integrable mappings. In Section 3 we use the processes of *reparametrization* and *replacement* [7, 8, 9] (terms introduced and defined in [9]) to construct three-dimensional measure-preserving integrable mappings.

2 Three-Dimensional Volume-Preserving Mappings

In this section we construct three-dimensional volume-preserving mappings (orientation-reversing and -preserving)[1] possessing two integrals, at least one of the integrals being quadratic in the three variables.

We begin with the orientation-reversing case. Consider the three-quadratic expression

$$I(x,y,z) = \sum A_{\alpha_1\beta_1\gamma_1}\ x^{\alpha_1}y^{\beta_1}z^{\gamma_1}, \quad (\alpha_1,\beta_1,\gamma_1 = 0,1,2), \tag{1}$$

where $A_{\alpha_1\beta_1\gamma_1}$ are independent parameters. Assume that (1) is invariant under a cyclic permutation of variables[2], i.e. $I(x,y,z) = I(y,z,x)$, and that the mapping, L, preserving $I(x,y,z)$ is reversible, i.e.

$$L \circ G \circ L = G, \tag{2}$$

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‡email: A.Iatrou@latrobe.edu.au apostolosiatrou@hotmail.com

[1]A mappings L is orientation reversing (orientation preserving) if $\det dL = -1$ ($\det dL = 1$).

[2]This guarantees that the mapping preserving this integral takes the form $x' = y, y' = z, z' = F(x,y,z)$, where F is some function.

with reversing symmetry $G : x' = y, y' = x, z' = x$. The reversing symmetry also implies that $I(x, y, z) = I(z, y, x)$. Under these conditions we obtain the integral

$$\begin{aligned} I(x,y,z) &= A_1 x^2y^2z^2 + A_2 xyz(xy+xy+yz) + A_3(x^2y^2+x^2z^2+y^2z^2) \\ &\quad + A_4 xyz(x+y+z) + A_5(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2) \\ &\quad + A_6 xyz + A_7(x^2+y^2+z^2) + A_8(xy+xz+yz) + A_9(x+y+z). \end{aligned} \tag{3}$$

The mapping, L, which leaves the integral (3) invariant can be derived by setting $x' = x$ and $y' = y$ and differencing the integral (3), i.e. $I(x', y', z') = I(x, y, z') = I(x, y, z)$. Then assuming $z' \neq z$, we can solve for z', to obtain the involution L_z. Finally, composing L_z with the cyclic shift $L_c : x' = y, y' = z, z' = x$, i.e. $L = L_z \circ L_c$, we obtain the non-trivial volume-preserving orientation-reversing mapping, L,

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{A_2y^2z^2 + A_4yz(y+z) + A_5(y^2+z^2) + A_6yz + A_8(y+z) + A_9}{A_1y^2z^2 + A_2yz(y+z) + A_3(y^2+z^2) + A_4yz + A_5(y+z) + A_7}. \end{aligned} \tag{4}$$

For a slightly more general map see [3].

Next, assume that the mapping (4) has a second integral with the following ansatz[3]

$$I(x,y,z) = \sum A_{\alpha_2\beta_2\gamma_2}\, x^{\alpha_2}y^{\beta_2}z^{\gamma_2}, \quad (\alpha_2, \gamma_2 = 0, 1, 2\,, \;\; \beta_2 = 0, 1, 2, 3, 4). \tag{5}$$

where $A_{\alpha_2\beta_2\gamma_2}$ are independent parameters. As the mapping (4) is reversible we also have $I_2(x, y, z) = I_2(z, y, x)$. We have found the following mappings which simultaneously preserve integrals of the form (3) and (5) :

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{\beta(y+z)^2 + \epsilon(y+z) + \xi}{\beta(y+z) + \gamma} \end{aligned} \tag{6}$$

with integrals

$$\begin{aligned} I_1 &= \beta(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz) + \gamma(x^2+y^2+z^2) \\ &\quad + \epsilon(xy+xz+yz) + \xi(x+y+z) \end{aligned} \tag{7}$$

$$\begin{aligned} I_2 &= \beta^2(x+y)^2(y+z)^2 + \beta(\epsilon-\gamma)[(x+y)^2(y+z) + (x+y)(y+z)^2] \\ &\quad + \gamma(\epsilon-2\gamma)[(x+y)^2 + (y+z)^2] + [\beta\xi + (\epsilon-\gamma)(\epsilon-2\gamma)](x+y)(y+z) \\ &\quad + \xi(\epsilon-2\gamma)[(x+y) + (y+z)] \end{aligned} \tag{8}$$

and

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{2\beta yz + \epsilon(y+z) + \xi}{\alpha yz + \beta(y+z) + \gamma} \end{aligned} \tag{9}$$

with integrals

$$\begin{aligned} I_1 &= \alpha^2x^2y^2z^2 + 2\alpha\beta xyz(xy+xz+yz) + \alpha\epsilon xyz(x+y+z) \\ &\quad + \beta^2(xy+xz+yz)^2 + \beta\epsilon(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 4xyz) \\ &\quad + (\alpha\xi - 2\beta\gamma)xyz + \gamma\epsilon(x^2+y^2+z^2 - xy - xz - yz) - \gamma^2(x^2+y^2+z^2) \\ &\quad + (\beta\xi + \epsilon^2)(xy+xz+yz) + \xi(\epsilon-\gamma)(x+y+z) \end{aligned} \tag{10}$$

$$\begin{aligned} I_2 &= (\beta+\epsilon)[\alpha xyz(x-y+z) + \gamma(x^2+y^2+z^2 - xy - yz + xz) + \xi(x+z)] \\ &\quad + [\beta^2(x+z) + \beta\epsilon(x+z+1) + \epsilon^2][xy + xz + yz - y^2]. \end{aligned} \tag{11}$$

[3]See the Appendix for the reason why the ansatz has this form.

We next consider the orientation-preserving case. Consider the three-quadratic expression (1) possessing the symmetry $I(x,y,z) = I(y,z,-x)$ [4]. Following the procedure outlined above we obtain the mapping

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= x + \frac{A(y-z)}{Byz+C} \end{aligned} \tag{12}$$

with integrals

$$\begin{aligned} I_1 &= B^2x^2y^2z^2 - ABxyz(x-y+z) - A(A+C)(xy-xz+yz) \\ &\quad - C(A+C)(x^2+y^2+z^2) \end{aligned} \tag{13}$$

$$\begin{aligned} I_2 &= B^3x^2y^2z^2 - C^3(xy+xz+yz) + ABC(2xz-2xy^2z-y) + A^2(B-C)y^2 \\ &\quad - BC^2(x^2+y^2+z^2+x+y+z+xyz[x+y+z]) \\ &\quad + 2AB^2xy^2z - B^2Cxyz(xyz+1) - AC^2(x+y)(y+z) \end{aligned} \tag{14}$$

We close this section with the following remark.

Remark The mapping (6) under the coordinate transformation $X = x+y, Y = y+z$ can be reduced to a two-dimensional area-preserving mapping, i.e.

$$L_1: \quad X' = Y\,, \quad Y' = -X - \frac{(\epsilon-\gamma)Y+\xi}{\beta Y+\gamma} \tag{15}$$

with the second integral, I_2, becoming

$$\begin{aligned} I &= \beta^2X^2Y^2 + \beta(\epsilon-\gamma)(X^2Y+XY^2) + \gamma(\epsilon-2\gamma)(X^2+Y^2) \\ &\quad + [\beta\xi + (\epsilon-\gamma)(\epsilon-2\gamma)]XY + \xi(\epsilon-2\gamma)(X+Y). \end{aligned} \tag{16}$$

Note that the first integral, I_1, does not reduce under this transformation.

In fact, the mapping (6) is a member of a recently-discovered hierarchy of integrable mappings given in [6], the three-dimensional asymmetric mapping[5] being

$$\begin{aligned} x' &= -x - \frac{\beta(y+z)^2 + \epsilon(y+z) + \xi_0}{\beta(y+z)+\gamma_0} \\ y' &= -y - \frac{\beta(x'+z)^2 + \epsilon(x'+z) + \xi_1}{\beta(x'+z)+\gamma_1} \\ z' &= -z - \frac{\beta(x'+y')^2 + \epsilon(x'+y') + \xi_2}{\beta(x'+y')+\gamma_2}. \end{aligned} \tag{17}$$

3 Three-Dimensional Measure-Preserving Mappings

In this section we apply the processes of reparametrization and replacement to the three-dimensional volume-preserving integrable mappings constructed above to construct measure-preserving integrable mappings. These examples illustrate how integrable three-dimensional families of mappings can be embedded in larger (i.e. higher number of parameters) integrable three-dimensional families of mappings via the processes of reparametrization and replacement.

Consider the mapping (9) when $\alpha = 0$ and $\epsilon = \gamma$, i.e.

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{2\beta yz + \gamma(y+z) + \xi}{\beta(y+z)+\gamma} \end{aligned} \tag{18}$$

[4] This symmetry is due to [3].

[5] In [6] this asymmetric mapping was shown to be obtained as a composition of three involutions. We believe that this guarantees the reversibility of this mapping. It seems, more generally, that a mapping obtained from a composition of involutions is reversible, see [6] for examples of such mappings.

which has integrals

$$\begin{aligned} I_1 &= \beta(xy+xz+yz)^2+\gamma(x+y)(x+z)(y+z)+\xi(xy+xz+yz) && (19)\\ I_2 &= (\beta+\gamma)[\gamma(x^2+y^2+z^2-xy-yz+xz)+\xi(x+z)] \\ &\quad +[\beta^2(x+z)+\beta\gamma(x+z+1)+\gamma^2][xy+xz+yz-y^2]. && (20)\end{aligned}$$

Notice that the parameters β, γ and ξ now appear linearly in the integral I_1. Reparametrizing the parameters, i.e. $\beta \to \beta_0+\beta_1 K, \gamma \to \gamma_0+\gamma_1 K, \xi \to \xi_0+\xi_1 K$ and the integral $I_1 \to \bar{I}_1 = I_1+\mu_0+\mu_1 K$, we obtain the mapping

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x-\frac{2(\beta_0+\beta_1 K)yz+(\gamma_0+\gamma_1 K)(y+z)+\xi_0+\xi_1 K}{(\beta_0+\beta_1 K)(y+z)+(\gamma_0+\gamma_1 K)}. && (21)\end{aligned}$$

Using $\bar{I}_1(x,y,z)=0$ (as $\bar{I}_1(x,y,z)=0 \Rightarrow \bar{I}_1(x',y',z')=0$) a new integral $K=k(x,y,z)$ can be defined. Define the map L_K to be the map (21) with replacement $K=k(x,y,z)$. The map L_K has two integrals $k(x,y,z)$ and $\bar{I}_2 = I_2(x,y,z)|_{K=k(x,y,z)}$, i.e.

$$\begin{aligned} k &= -\frac{\beta_0(xy+xz+yz)^2+\gamma_0(x+y)(x+z)(y+z)+\xi_0(xy+xz+yz)+\mu_0}{\beta_1(xy+xz+yz)^2+\gamma_1(x+y)(x+z)(y+z)+\xi_1(xy+xz+yz)+\mu_1} && (22)\\ \bar{I}_2 &= \{[\beta_0+\gamma_0+(\beta_1+\gamma_1)K][(\gamma_0+\gamma_1 K)(x^2+y^2+z^2-xy-yz+xz) \\ &\quad +(\xi_0+\xi_1 K)(x+z)]+[(\beta_0+\beta_1 K)^2(x+z) \\ &\quad +(\beta_0+\beta_1 K)(\gamma_0+\gamma_1 K)(x+z+1) \\ &\quad +(\gamma_0+\gamma_1 K)^2][xy+xz+yz-y^2]\}|_{K=k(x,y,z)}. && (23)\end{aligned}$$

The map L_K is also measure preserving with

$$m(x,y,z) = \left[\frac{\partial \bar{I}_1}{\partial K}\right]^{-1}. \qquad (24)$$

Consider the mapping (9) when $\alpha=0$ and $\epsilon=\gamma=\beta$, i.e.

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x-\frac{\beta(2yz+y+z)+\xi}{\beta(y+z+1)} && (25)\end{aligned}$$

which has integrals

$$\begin{aligned} I_1 &= \beta[(xy+xz+yz)^2+(x+y)(x+z)(y+z)]+\xi(xy+xz+yz) && (26)\\ I_2 &= \beta(x+z)(x+z+xy+xz+yz-y^2)+\xi(x+z). && (27)\end{aligned}$$

Notice that the parameters β and ξ now appear linearly in both integrals. Reparametrizing the parameters and the integrals, i.e. $\beta \to \beta_0+\beta_1 K_1+\beta_2 K_2$, $\xi \to \xi_0+\xi_1 K_1+\xi_2 K_2$, $I_1 \to \bar{I}_1 = I_1+\mu_0+\mu_1 K_1+\mu_2 K_2$ and $I_2 \to \bar{I}_2 = I_2+\nu_0+\nu_1 K_1+\nu_2 K_2$, we obtain the mapping

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -x-\frac{(\beta_0+\beta_1 K_1+\beta_2 K_2)(2yz+y+z)+\xi_0+\xi_1 K_1+\xi_2 K_2}{(\beta_0+\beta_1 K_1+\beta_2 K_2)(y+z+1)}, && (28)\end{aligned}$$

with integrals $\bar{I}_1(x,y,z)$ and $\bar{I}_2(x,y,z)$. Setting $\bar{I}_1(x,y,z)=0$ and $\bar{I}_2(x,y,z)=0$ it is possible to solve for $K_1=k_1(x,y,z)$ and $K_2=k_2(x,y,z)$ as $\bar{I}_1$ and $\bar{I}_2$ are linear in K_1 and K_2. Define the map $L_{K_1 K_2}$

to be the map (28) with replacements $K_1 = k_1(x,y,z)$ and $K_2 = k_2(x,y,z)$. The map $L_{K_1K_2}$ has the integrals $K_1 = k_1(x,y,z)$ and $K_2 = k_2(x,y,z)$. The map $L_{K_1K_2}$ is also measure preserving with

$$m(x,y,z) = \begin{vmatrix} \frac{\partial \bar{I}_1}{\partial K_1} & \frac{\partial \bar{I}_1}{\partial K_2} \\ \frac{\partial \bar{I}_2}{\partial K_1} & \frac{\partial \bar{I}_2}{\partial K_2} \end{vmatrix}^{-1} . \tag{29}$$

The integrals to the above maps can be shown to be functionally independent and in involution with respect to the following Poisson structure [1]

$$m(x,y,z) \begin{pmatrix} 0 & \frac{\partial I}{\partial z} & -\frac{\partial I}{\partial y} \\ -\frac{\partial I}{\partial z} & 0 & \frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial y} & -\frac{\partial I}{\partial x} & 0 \end{pmatrix}, \tag{30}$$

where I is either one of integrals and $m(x,y,z)$ is the measure.

Finally, we consider the mapping (6). As noted in the remark at the end of Section 2 we can use a coordinate transformation, i.e. $X = x+y$ and $Y = y+z$, to reduce the mapping to a two-dimensional mapping. Importantly, however, the three-quadratic integral, I_1, does not reduce under this coordinate transformation and as a result if we use the processes of reparametrisation and replacement on this integral then the resulting mapping, L_r, is not reducible to a two dimensional mapping, although for every fixed K it is. The remark at the end of Section 2 also shows that the reduced mapping has a biquadratic integral and thus can be explicitly integrated[6], see [7]. This result can be used to integrate the mapping L_r also *but* this time curve-wise (leaf-wise).[7]

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Appendix

Our work on multidimensional integrable mappings (particularly [6]) has lead us to the following observation about the degree of the variables that occur in the integrals :

A 2n-dimensional volume-preserving integrable mapping with variables $v_0, \ldots, v_{2n-1}$*, which has one n-quadratic integral, possesses an additional* $(n-1)$ *integrals of the following form*

$$\begin{aligned} I_2 &= \sum A_{\alpha_{21}\ldots\alpha_{2,2n}} v_0^{\alpha_{21}} \ldots v_{2n-1}^{\alpha_{2,2n}}, \quad (\alpha_{21}, \ldots, \alpha_{2,2n} = 0, \ldots, 4) \\ &\vdots \\ I_n &= \sum A_{\alpha_{n,1}\ldots\alpha_{n,2n}} v_0^{\alpha_{n,1}} \ldots v_{2n-1}^{\alpha_{n,2n}}, \quad (\alpha_{n,1}, \ldots, \alpha_{n,2n} = 0, \ldots, 2n). \end{aligned} \tag{31}$$

While a (2n+1)-dimensional volume-preserving integrable mapping with variables $v_0, \ldots, v_{2n}$*, which has one n-quadratic integral, possesses an additional n integrals of the following form*

$$\begin{aligned} I_2 &= \sum A_{\alpha_{21}\ldots\alpha_{2,2n+1}} v_0^{\alpha_{21}} \ldots v_{2n}^{\alpha_{2,2n+1}}, \quad (\alpha_{21}, \ldots, \alpha_{2,2n+1} = 0, \ldots, 4) \\ &\vdots \\ I_{n+1} &= \sum A_{\alpha_{n+1,1}\ldots\alpha_{n+1,2n+1}} v_0^{\alpha_{n+1,1}} \ldots v_{2n}^{\alpha_{n+1,2n+1}}, \quad (\alpha_{n+1,1}, \ldots, \alpha_{n+1,2n+1} = 0, \ldots, 2n+2). \end{aligned} \tag{32}$$

In the case we have considered in this paper, the power of the first and last variables ranges from 0 to 2.

[6] This also is true for its asymmetric form.

[7] We believe that the maps considered in Case 2 of [6, Section 3] can be integrated in this way also.

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